

Supersymmetric quantum mechanical generalized MIC-Kepler system

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Abstract

We construct supersymmetric generalized MIC-Kepler system and show that the systems with half integral Dirac quantization condition $\mu = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \dots$ belong to a supersymmetric family (hierarchy of Hamiltonians) with same spectrum between the respective partner Hamiltonians except for the ground state. Similarly the systems with integral Dirac quantization condition $\mu = \pm 1, \pm 2, \pm 3, \dots$ belong to another family. We show that, it is necessary to introduce additional potential to MIC-Kepler system like generalized MIC-Kepler system in order to unify the two family into one. We also reproduce the results of the (supersymmetric) Hydrogenic problem in our study.

Keywords: Supersymmetric quantum mechanics, generalized MIC-Kepler system, bound states

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1 Introduction

Supersymmetry, a symmetry between fermions and bosons, was first introduced in High Energy Physics in order to obtain a unified description of all basic interactions of nature [1, 2, 3, 4, 5]. But fermions and bosons being particles with different properties, for example fermion obey Pauli exclusion principle but boson does not, supersymmetry is a highly nontrivial symmetry. In particle physics, supersymmetry predicts that corresponding to every basic constituent of nature, there should be a supersymmetric partner with spin differing by half-integral unit and it also predicts that the supersymmetric partners must have identical mass if supersymmetry is preserved. In unified theory for the basic interactions of nature, supersymmetry predicts the existence of supersymmetric partners of all the fundamental particles in nature (i.e., quarks, leptons, gluon, photon, etc.). But so far there is no experimental evidence that supersymmetry is preserved in nature. Then people started to think supersymmetry breaking and this inspired Witten [6] to study supersymmetry breaking in nonrelativistic quantum mechanics as a toy model for supersymmetry breaking in quantum field theory. Supersymmetry has application in atomic, nuclear, condensed matter and statistical physics [13] also.

Supersymmetric quantum mechanics has become a separate field of study and has got lot of interest for its beautiful mathematical insight as well as for various aspect of nonrelativistic quantum mechanics. Nonrelativistic coulomb problem is a standard problem in supersymmetric quantum mechanics, for example. In Ref. [8] it has been shown that the distinct spectrum of certain atoms and ions have phenomenological quantum mechanical supersymmetry. For example, the s levels of the lithium atom can be interpreted as the supersymmetric partner of the hydrogen atom s levels in the absence of electron-electron interactions and provided the valence electron is far enough removed from the core electrons. It is

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also discussed that the supersymmetry is broken if electron-electron interaction is present for the obvious reason that the level with fixed principal quantum number n and different orbital angular momentum quantum number l splits up. For other discussion on coulomb problem see Ref. [9]

MIC-Kepler system [10, 11] is very similar to the nonrelativistic coulomb problem, where two dyons with electric and magnetic charges e_1, g_1 and e_2, g_2 respectively are present. It received lot of interest because one can retain all the symmetry of nonrelativistic coulomb system like $O(4)$ for bound state and $O(1, 3)$ for continuum state if a potential of the form $\sim \frac{1}{r^2}$ is added to the Hamiltonian by hand. It has been further generalized [12] keeping the symmetry of the system undisturbed and the system is called generalized MIC-Kepler system for obvious reason. In this present work we will discuss about this generalized MIC-Kepler system [12] in the framework of supersymmetric quantum mechanics.

The paper is organized as follows: In Sec. 2, we recapitulate the basic formalism of supersymmetric quantum mechanics. In Sec. 3, we make the supersymmetric extension of the generalized MIC-Kepler system. We conclude in Sec. 4.

2 General review of the formalism of supersymmetric quantum mechanics

In this section we recapitulate the general feature of supersymmetric quantum mechanics. For detail review see [13] and the references therein. The supersymmetric quantum mechanical Hamiltonian can be written as

$$H_{susy}(x) = H_+(x) \oplus H_-(x). \quad (2.1)$$

In $\hbar = m = 1$ unit, the two partner Hamiltonians $H_+(x)$ and $H_-(x)$ take the form

$$H_{\pm}(x) = -\frac{1}{2} \frac{d^2}{dx^2} + V_{\pm}(x), \quad (2.2)$$

where the supersymmetric partner potentials $V_+(x)$ and $V_-(x)$ can be written in terms of superpotential $W(x)$ as

$$V_{\pm}(x) = W^2(x) \mp \frac{1}{\sqrt{2}} W'(x), \quad (2.3)$$

It can be easily checked from Eq. (2.2) Eq. (2.3) that the Hamiltonians $H_+(x)$ and $H_-(x)$ can be factorized in the form

$$H_+(x) = A(x)^\dagger A(x), \quad H_-(x) = A(x) A(x)^\dagger, \quad (2.4)$$

if we consider the first order differential operator $A(x)$ and $A(x)^\dagger$ of the form

$$A(x) = \frac{1}{\sqrt{2}} \frac{d}{dx} + W(x), \quad A^\dagger(x) = -\frac{1}{\sqrt{2}} \frac{d}{dx} + W(x). \quad (2.5)$$

We can define two supercharges $Q(x)$ and $Q(x)^\dagger$ of the form

$$Q(x) = A(x)\sigma_-, \quad Q^\dagger(x) = A^\dagger(x)\sigma_+, \quad (2.6)$$

where σ_+ and σ_- can be written in terms of Pauli matrices as

$$\sigma_{\pm} = \frac{1}{2}(\sigma_x \pm i\sigma_y). \quad (2.7)$$

The Hamiltonian $H_{susy}(x)$ together with the two supercharge $Q(x)$ and $Q^\dagger(x)$ form the the closed superalgebra $sl(1/1)$:

$$\begin{aligned} [H_{susy}(x), Q(x)] &= [H_{susy}(x), Q^\dagger(x)] = 0, \quad \{Q(x), Q^\dagger(x)\} = H_{susy}(x), \\ \{Q(x), Q(x)\} &= \{Q^\dagger(x), Q^\dagger(x)\} = 0. \end{aligned} \quad (2.8)$$

The energy eigenvalues, the wave functions and the S-matrices of $H_+(x)$ and $H_-(x)$ are related. In particular, if E is the eigenvalue of $H_+(x)$ then it is also the eigenvalue of $H_-(x)$ and vice a versa. The energy eigenvalues of both $H_+(x)$ and $H_-(x)$ are positive semi-definite ($E_n^{(+,-)} \geq 0$). The Schrödinger equation for $H_+(x)$ takes the form

$$H_+(x)\psi_n^+(x) = A^\dagger(x)A(x)\psi_n^+(x) = E_n^+\psi_n^+(x). \quad (2.9)$$

On multiplying both sides of this equation by the operator A from the left, we get

$$H_-(x)(A(x)\psi_n^+(x)) = A(x)A^\dagger(x)A(x)\psi_n^+(x) = E_n^+(A(x)\psi_n^+(x)). \quad (2.10)$$

Similarly, the Schrödinger equation for H_-

$$H_-(x)\psi_n^-(x) = A(x)A^\dagger(x)\psi_n^-(x) = E_n^-\psi_n^-(x) \quad (2.11)$$

implies

$$H_+(x)(A^\dagger(x)\psi_n^-(x)) = A^\dagger(x)A(x)A^\dagger(x)\psi_n^-(x) = E_n^-(A^\dagger(x)\psi_n^-(x)). \quad (2.12)$$

Thus if E is an eigenvalue of the Hamiltonian $H_+(x)/H_-(x)$ with eigenfunction $\psi(x)$, then same E is also the eigenvalue of the Hamiltonian $H_-(x)/H_+(x)$ and the corresponding eigenfunction is $A(x)\psi(x)/A^\dagger(x)\psi(x)$.

The above proof breaks down in case $A(x)\psi_0^+(x) = 0$, i.e. when the ground state is annihilated by the operator $A(x)$. Thus the exact relationship between the eigenstates of the two Hamiltonians will crucially depend on if $A(x)\psi_0^+(x)$ is zero or nonzero, i.e. if the ground state energy E_0^+ is zero or nonzero.

For the case $A\psi_0^+ \neq 0$ the proof goes through for all the states including the ground state and hence all the eigenstates of the two Hamiltonians are paired, i.e. they are related by ($n = 0, 1, 2, \dots$)

$$E_n^- = E_n^+ > 0, \quad \psi_n^-(x) = [E_n^+]^{-1/2}A(x)\psi_n^+(x), \quad \psi_n^+(x) = [E_n^-]^{-1/2}A^\dagger(x)\psi_n^-(x). \quad (2.13)$$

For the case $A\psi_0^+ = 0$, $E_0^+ = 0$ and this state is unpaired while all other states of the two Hamiltonian are paired. It is then clear that the eigenvalues and eigenfunctions of the two Hamiltonians $H_+(x)$ and $H_-(x)$ are related by ($n = 0, 1, 2, \dots$)

$$E_n^- = E_{n+1}^+, \quad E_0^+ = 0, \quad \psi_n^-(x) = [E_{n+1}^+]^{-1/2}A(x)\psi_{n+1}^+(x), \quad \psi_{n+1}^+(x) = [E_n^-]^{-1/2}A^\dagger(x)\psi_n^-(x). \quad (2.14)$$

$A(x)\psi_0^+(x) = 0$ can be interpreted in the following two different ways depending on whether the superpotential $W(x)$ or the ground state wave function $\psi_0^+(x)$ is known. In case $W(x)$ is known then one can solve the equation $A(x)\psi_0^+ = 0$ and the ground state wavefunction of $H_+(x)$ is given, in terms of the superpotential $W(x)$ by

$$\psi_0^+(x) \sim \exp[-\sqrt{2} \int^x W(y)dy]. \quad (2.15)$$

Instead, if $\psi_0^+(x)$ is known, then this equation gives us the superpotential $W(x)$, i.e.

$$W(x) = -2^{-\frac{1}{2}}\psi_0'^+(x)(\psi_0^+(x))^{-1} \quad (2.16)$$

The above procedure can in fact be repeatedly used to generate a hierarchy of Hamiltonians [13]. For example if the original Hamiltonian H_1 has $p(\geq 1)$ bound states with eigenvalues $E_n^{(1)}$ and eigenfunctions $\psi_n^{(1)}$ with $0 \leq n \leq (p-1)$, then a hierarchy of $(p-1)$ Hamiltonians H_2, H_3, \dots, H_p can be constructed such that the m 'th member of the hierarchy of Hamiltonians (H_m) has the same eigenvalue as H_1 except that the first $(m-1)$ eigenvalues of H_1 are missing in (H_m) .

3 Supersymmetric generalized MIC-Kepler System

We now come to the discussion of supersymmetric extension of generalized MIC-Kepler [12] system, which is our interest in this work. The generalized MIC-Kepler system, in a system of units $\hbar = c = e = 1$, with mass taken to be unity, is given by the Hamiltonian [12]

$$H = \frac{1}{2}(-i\nabla - \mu\mathbf{A})^2 + \frac{\mu^2}{2r^2} - \frac{1}{r} + \frac{c_1}{r^2(1 + \cos\theta)} + \frac{c_2}{r^2(1 - \cos\theta)}, \quad (3.1)$$

Here \mathbf{A} is the magnetic vector potential of the Dirac monopole, given by

$$\mathbf{A} = -\frac{\sin\theta}{r(1 - \cos\theta)}\hat{\phi}, \quad (3.2)$$

such that $\text{curl}\mathbf{A} = \frac{\mathbf{r}}{r^3}$. c_1, c_2 are nonnegative constants and μ , the Dirac quantization condition, takes the values $0, \pm\frac{1}{2}, \pm 1, \dots$.

The technique of supersymmetric quantum mechanics can be applied to this system if the eigenvalue problem

$$H\Psi(r, \theta, \phi) = E\Psi(r, \theta, \phi) \quad (3.3)$$

can be separated into one dimensional radial equation. This system can indeed be separated into radial coordinate and the radial equation is given by

$$-\frac{1}{2}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) + \left[-\frac{1}{r} - E + \frac{(j + \frac{\delta_1 + \delta_2}{2})(j + \frac{\delta_1 + \delta_2}{2} + 1)}{2r^2}\right]R(r) = 0 \quad (3.4)$$

where

$$\begin{aligned} \delta_1 &= \sqrt{(m-s)^2 + 4c_1} - |m-s|, \\ \delta_2 &= \sqrt{(m+s)^2 + 4c_2} - |m+s|, \end{aligned} \quad (3.5)$$

and j takes the values

$$j = |s|, |s| + 1, |s| + 2, \dots \quad (3.6)$$

The first order derivative term of Eq. (3.4) can be removed by the transformation $R(r) \rightarrow \chi(r)/r$ and the resulting form is given by

$$-\frac{1}{2}\frac{d^2\chi(r)}{dr^2} + \left[-\frac{1}{r} - E + \frac{(j + \frac{\delta_1 + \delta_2}{2})(j + \frac{\delta_1 + \delta_2}{2} + 1)}{2r^2}\right]\chi(r) = 0 \quad (3.7)$$

This is similar to one dimensional Schrödinger equation and therefore subjected to supersymmetric treatment.

Now we can construct the bosonic radial Hamiltonian $H_+(r)$ for the generalized MIC-Kepler system from Eq. (3.7) of the form

$$H_+(r) = -\frac{1}{2}\frac{d^2}{dr^2} + V_+(r) \quad (3.8)$$

with $V_+(r)$ given by

$$V_+(r) = -\frac{1}{r} + \frac{(j + \frac{\delta_1 + \delta_2}{2})(j + \frac{\delta_1 + \delta_2}{2} + 1)}{2r^2} + \frac{1}{2(j + \frac{\delta_1 + \delta_2}{2} + 1)^2} \quad (3.9)$$

The energy eigenvalue of the Hamiltonian (3.8) takes the form

$$E_n^+ = \frac{1}{2(j + \frac{\delta_1 + \delta_2}{2} + 1)^2} - \frac{1}{2(n + \frac{\delta_1 + \delta_2}{2})^2} \quad \text{for, } n \geq j + 1 \quad (3.10)$$

From Eq. (2.3) and Eq. (3.9) we can work out the superpotential $W(r)$ as

$$W(r) = \frac{1}{j + \frac{\delta_1 + \delta_2}{2} + 1} - \frac{j + \frac{\delta_1 + \delta_2}{2} + 1}{r} \quad (3.11)$$

From Eq. (2.3) and Eq. (3.11) we can calculate the supersymmetric partner to V_+ is

$$V_-(r) = -\frac{1}{r} + \frac{(j + \frac{\delta_1 + \delta_2}{2} + 1)(j + \frac{\delta_1 + \delta_2}{2} + 2)}{2r^2} + \frac{1}{2(j + \frac{\delta_1 + \delta_2}{2} + 1)^2} \quad (3.12)$$

Now some important observations can be made in addition to the nonrelativistic coulomb problem discussed in [8, 9].

3.1 Observation 1

Consider the situation $c_1 = c_2 = \mu = 0$ in the Hamiltonian Eq. (3.1). This is the Hydrogen atom problem. The potential $V_+(r)$ takes the form,

$$V_+(r) = -\frac{1}{r} + \frac{l(l+1)}{2r^2} + \frac{1}{2(l+1)^2} \quad (3.13)$$

and the corresponding eigenvalue is given by

$$E_n^+ = \frac{1}{2(l+1)^2} - \frac{1}{2(n)^2} \quad \text{for, } n \geq l + 1 \quad (3.14)$$

Since the result Eq. (3.14) is already calculated in Ref. [8, 9], we are not discussing the physical aspect of the result here.

3.2 Observation 2

Consider the situation $c_1 = c_2 = 0$ in the Hamiltonian Eq. (3.1). This is the two particle problem which have magnetic charges g_1, g_2 in addition to their electric charges e_1, e_2 . The potential $V_+(r)$ takes the form,

$$V_+(r) = -\frac{1}{r} + \frac{j(j+1)}{2r^2} + \frac{1}{2(j+1)^2} \quad (3.15)$$

and the corresponding eigenvalue is given by

$$E_n^+ = \frac{1}{2(j+1)^2} - \frac{1}{2(n)^2} \quad \text{for, } n \geq j + 1 \quad (3.16)$$

It is to be noted that the expression looks like Hydrogenic problem Eq. (3.13) and Eq. (3.14), but it is not the same problem because j can take values $j = |\mu|, |\mu| + 1, |\mu| + 2, \dots$, whereas for Hydrogenic problem l can take non negative integral values only. If one considers the case of lowest value of j , i.e., $j = |\mu|$, then it is easy to show that the systems with Dirac quantization condition differing by $|\mu_1| - |\mu_2| = 1$ are supersymmetric partners. Since μ can takes values like $\mu = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots$, it is straightforward to conclude that system with $\mu = \pm\frac{1}{2}, \pm\frac{3}{2}, \dots$ belong to one supersymmetric family and system with $\mu = 0, \pm 1, \pm 2, \dots$ belong to other supersymmetric family. The two family gets decoupled in this scenario. In the next subsection we will show that it is necessary to incorporate the additional potentials with the MIC-Kepler system, as it has been done in generalized MIC-Kepler system, to get a unified supersymmetric family.

3.3 Observation 3

Finally, consider the situation when all the three parameters μ, c_1, c_2 are nonzero. The potential $V_+(r)$ and the respective energy eigenvalue E_n^+ is given by Eq. (3.9) and Eq. (3.10) respectively. Now consider the situation when $\delta_1 + \delta_2 = 1$. This can be achieved by appropriately tuning the constant parameter c_1 and c_2 in the Hamiltonian (3.1). Replacing $\delta_1 + \delta_2 = 1$ in Eq. (3.9) and Eq. (3.10), we can see that systems with half integral Dirac quantization condition, $\mu = \pm\frac{1}{2}, \pm\frac{3}{2}, \dots$, belong to the family of integral Dirac quantization condition of subsection (3.2)

4 Conclusion

Supersymmetry has been shown to be a good symmetry for nonrelativistic coulomb system [8, 9]. It has been successfully applied to many other systems, for example harmonic oscillator system. We show in our calculation that if supersymmetry is conserved in MIC-Kepler system, then there exist two kind of supersymmetric family, one with half integral Dirac quantization condition $\mu = \pm\frac{1}{2}, \pm\frac{3}{2}, \dots$ and other with integral Dirac quantization condition $\mu = 0, \pm 1, \pm 2, \dots$. We also show that in order to unify this two kind of supersymmetric family into one, we need to generalize MIC-Kepler system like Ref. [12]. Two extra potential terms with coefficients c_1 and c_2 in the Hamiltonian (3.1) allows us to unify the apparently separated supersymmetric family. We reproduce the nonrelativistic coulomb result [8, 9] in our formulation in subsection (3.1). Supersymmetry, supersymmetry breaking and self-adjointness is also an interesting issue [14]. We hope to discuss this for generalized MIC-Kepler system in future.

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